New approaches to solving the Saint-Venant equations

A virtual presentation for River Flow 2020

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Introduction

The paper submitted for River Flow2 2020 covers several advances in differential and finite-volume forms for the Saint-Venant equations.

Herein we'll just cover issues associated with splitting the piezometric pressure gradient into a body force and a pressure-like force.

The differential equations of Saint-Venant for momentum

Saint-Venant's form

$$\frac{\partial Q}{dt} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -gAS_f$$

Common form with S_0 for bottom slope

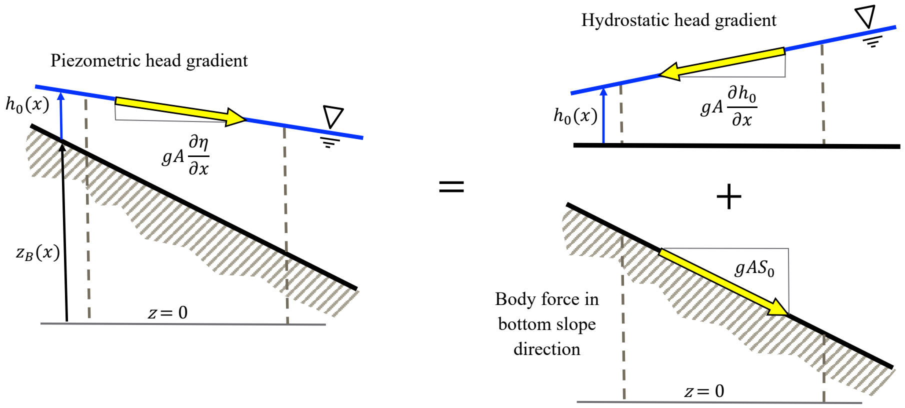
$$\frac{\partial Q}{dt} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial h_0}{\partial x} = gA\left(S_0 - S_f\right)$$

where:

- Q is the flow rate
- A is the cross sectional area
- η is the water surface elevation (Piezometric head)
- $S_f = f(Q^2, A, ...)$ is the friction slope (positive downstream)
- S_0 is a source term affected by non-uniform geometry.
- h_0 is the local hydrostatic head

The difference in forms is splitting η

$$\eta \equiv h_0 + z_0 \quad \rightarrow \quad \frac{\partial \eta}{\partial x} = \frac{\partial h_0}{\partial x} - S_0$$



Mathematically equivalent forms, but not numerically

Saint-Venant's form has a smooth RHS if η and S_f in source term are smooth:

$$\frac{\partial Q}{dt} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = gA\left(-\frac{\partial \eta}{\partial x} - S_f\right)$$

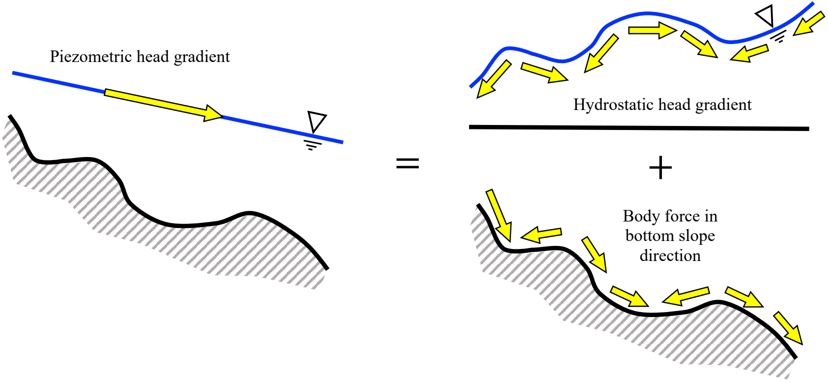
Bottom slope form is only smooth for non-smooth S_0 if $\partial h_0/\partial x$ exactly balances S_0

$$\frac{\partial Q}{dt} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) = gA\left(-\frac{\partial h_0}{\partial x} + S_0 - S_f\right)$$

This is known as the "well-balanced" problem.
Failure to have a well-balanced source term causes un-physical behaviour and numerical instabilities.

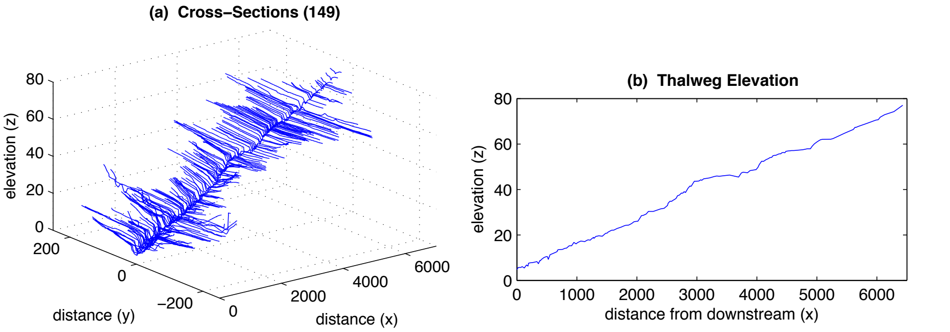
What it means to require balance of S_0 and $\partial h_0/\partial x$

Smooth solution of η becomes a non-smooth solution of h_0



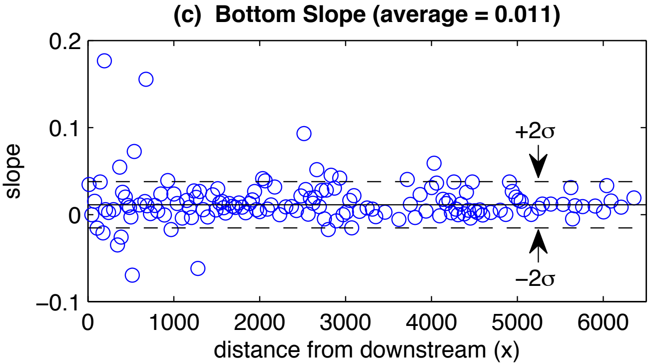
S_0 is effectively a body force that can change direction sharply in space, which is destabilizing to a SV solver.

Why do we care about S_0? Example of Waller Creek, Texas, USA



High resolution data set for urban creek.

At fine resolution, S_0 is highly variable



Do we really have to use $S_0(x)$?

S_0 is arbitrary based on using the bottom elevation $z_0(x)$

$$\eta = h_0 + z_0 \quad \rightarrow \quad \frac{\partial \eta}{\partial x} = \frac{\partial h_0}{\partial x} + \frac{\partial z_0}{\partial x}$$

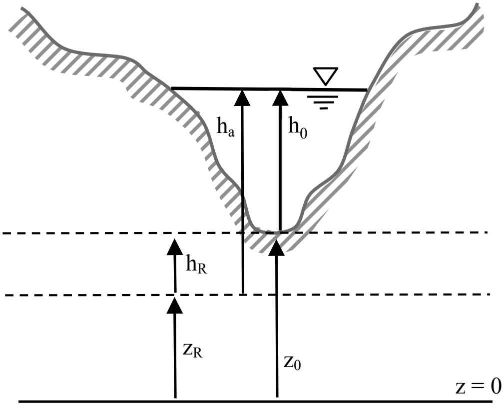
Define

$$S_0 \equiv -\frac{\partial z_0}{\partial x}$$

It follows that

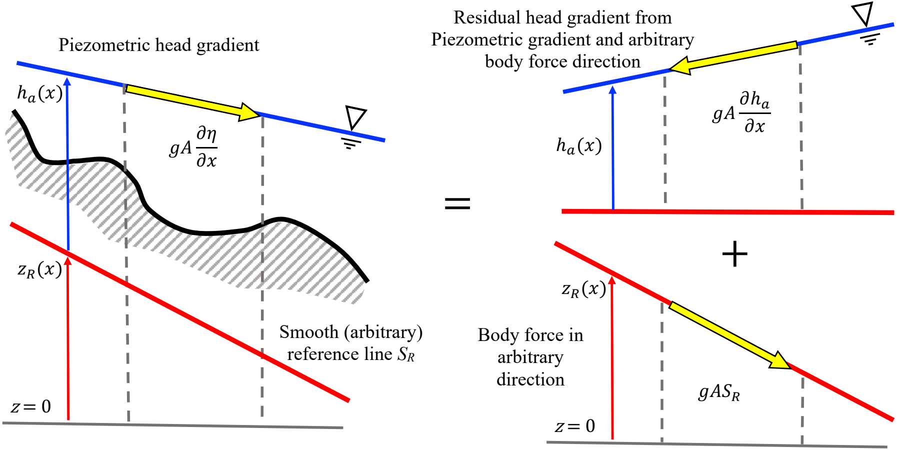
$$\frac{\partial \eta}{\partial x} = \frac{\partial h_0}{\partial x} - S_0$$

Redefinition: $\eta = h_0 + z_0 \quad = \quad h_a + z_R$

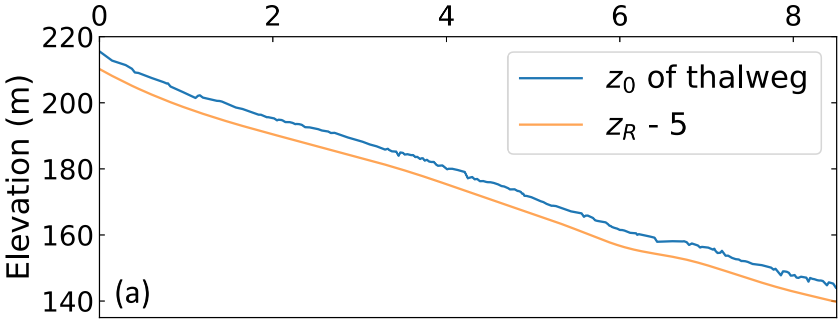


$z_R(x)$ is arbitrary and may be chosen for smoothness (e.g., approximate spline fit)

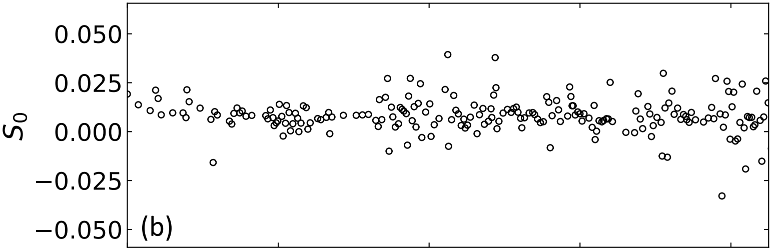
$z_R \to S_R$ as smooth body force



Waller Creek comparing z_0 and z_R

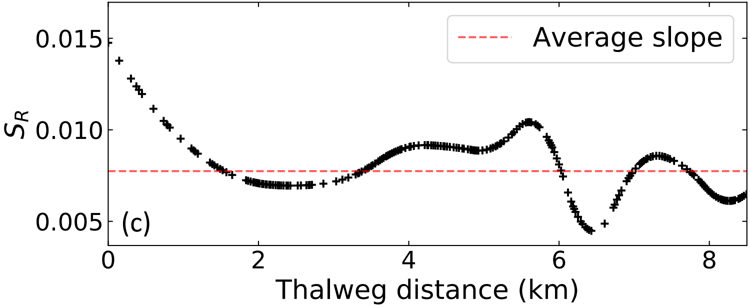


Waller Creek S_0 computed from z_0



S_0 range is roughly -1.5×10^{-2} to $+2.5 \times 10^{-2}$ plus outliers

Waller Creek $S_R(x)$ computed from approximating spline $z_R(x)$



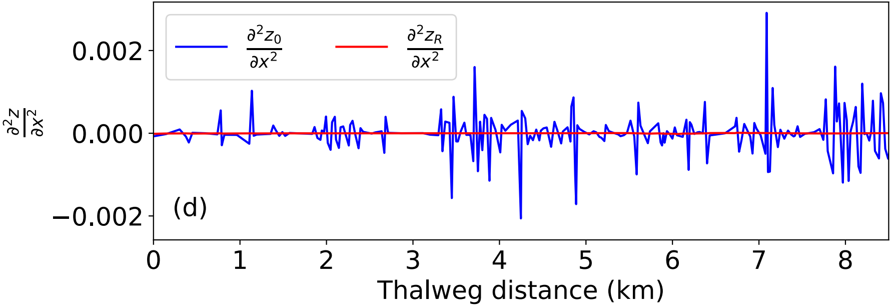
S_R range is roughly -0.5×10^{-2} to $+1.5 \times 10^{-2}$ without outliers

Compared to

S_0 range is roughly -1.5×10^{-2} to $+2.5 \times 10^{-2}$ plus outliers

Waller Creek gradients of S_0 and S_R

The curvature of the reference lines z_0 vs. z_R



Abrupt changes in $S_0(x)$ are likely to cause convergence problems.
There are no abrupt changes in $S_R(x)$.

The differential equations of Saint-Venant for momentum

Saint-Venant's form

$$\frac{\partial Q}{dt} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} = -gAS_f$$

Common form with S_0 for real bottom slope

$$\frac{\partial Q}{dt} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial h_0}{\partial x} = gA\left(S_0 - S_f\right)$$

New form with S_R for smooth reference slope

$$\frac{\partial Q}{dt} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial h_a}{\partial x} = gA\left(S_R - S_f\right)$$

Key points

- Traditional bottom slope (S_0) is arbitrary.
- Replace with a smooth reference slope (S_R).
- Smooth S_R does not smooth the actual geometry.
- Can be used with either finite difference or finite volume.
- Unlike h_0, the associated depth (h_a) does not represent true hydrostatic head, but we can easily recover that value if needed.
- Method ensures that slope in source term is smooth!

Other new advances (references on last slide)

Integration of piezometric pressure over bathymetry is the key complexity in 1D Saint-Venant equation (Hodges 2019).

A new derivation of the finite-volume SVE produces a piezometric pressure quadrature term that admits a range of conservative discretizations (Hodges 2019).

SVE with bottom slope is inherently well-balanced for any advective scheme (Yu et al., 2020; Hodges et al., 2020).

Important lesson: Don't use SVE forms that apply the bottom slope S_0 and the depth h_0 – they are inherently difficult to get well balanced (Yu et al., 2020; Hodges et al., 2020).

The new idea of *Timescale Interpolation* may have some advantages over spatial interpolation for reconstruction at coarse resolution (Hodges and Liu, 2019).

A new approach to handling transitions from free surface surcharged pipe has been developed (Hodges, 2020).

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References. Email hodges@utexas.edu (last slide)

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